

Best L_p Approximation*

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Abstract. A new algorithm is presented for the best L_p approximation of a continuous function over a discrete set or a finite interval with $2 < p < \infty$. Methods to accelerate the convergence of the Rice-Usow extension of Lawson's algorithm as well as the new algorithm are presented, and the result of a numerical example is given.

1. **Introduction.** In 1961 Lawson [1] gave an algorithm for finding the Chebyshev approximation on a finite point set. In 1968 Rice and Usow [2] extended the algorithm to the best L_p approximation with $2 < p < \infty$. These algorithms apply on a finite point set but not on a finite interval, and their convergences are slow.

In this paper, we describe a new algorithm for the best L_p approximation. It is an adaptation of Newton-Raphson's method, and is applicable both on a finite interval and on a finite point set with $2 < p < \infty$. The convergence of this iterative method is quadratic. We also give a scheme for accelerating the convergence of the extended Lawson algorithm as well as the new method. Finally, the result of a numerical example is given.

2. **Best L_p Approximation.** Let $f(x)$ be continuous in $[0, 1]$; let $\{\varphi_i(x)\}$, $i = 1, 2, \dots, n$, be a set of continuous and linearly independent functions in $[0, 1]$; let $L(A, x) = \sum_{i=1}^n a_i \varphi_i(x)$, where $A = (a_1, a_2, \dots, a_n)^T$; let $2 < p < \infty$; and let $w(x)$ be a nonnegative and Riemann integrable function in $[0, 1]$. In this paper, we say $L(A, x)$ is the best $L_{p,w}$ (or L_p) approximation of $f(x)$ in $[0, 1]$, if

$$\int_0^1 |f(x) - L(A, x)|^p w(x) dx \leq \int_0^1 |f(x) - L(B, x)|^p w(x) dx$$

for all $B = (b_1, b_2, \dots, b_n)^T$.

The new algorithm for the best L_p approximation of

$$(1) \quad F(A) = \int_0^1 |f(x) - L(A, x)|^p dx$$

is as follows:

Starting from the initial coefficient set A_0 , at the k th iteration

Step 1. Set $w_k(x) = |f(x) - L(A_{k-1}, x)|^{p-2}$.

Step 2. Find the least squares approximation $L(B_k, x)$ of the function $f(x)$ with the weight function $w_k(x)$.

Step 3. Set $A_k = \{(p-2)A_{k-1} + B_k\}/(p-1)$.

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We show below that the above algorithm is a rearranged form of Newton-Raphson's method. In view of the fact that (1) is a strictly convex function of A [3], it follows that the iteration always converges [4].

The minimization of (1) may be achieved by finding the roots of

$$(2) \quad \partial F(A)/\partial a_i = 0, \quad i = 1, 2, \dots, n,$$

whereas (2) can be solved iteratively by finding the solutions of

$$(3) \quad \frac{\partial F(A)}{\partial a_i} + \sum_{j=1}^n \frac{\partial^2 F(A)}{\partial a_i \partial a_j} \Delta a_j = 0$$

and replacing a_j with $a_j + \Delta a_j$ at each iteration, which is the Newton-Raphson method.

From (1) we find, at the k th iteration, that

$$(4) \quad \begin{aligned} \frac{\partial F(A_{k-1})}{\partial a_i} &= -p \int_0^1 |f(x) - L(A_{k-1}, x)|^{p-1} \operatorname{sgn}(\varphi_i(x), f(x) - L(A_{k-1}, x)) dx \\ &= -p \int_0^1 w_k(x)(f(x) - L(A_{k-1}, x))\varphi_i(x) dx \end{aligned}$$

and

$$(5) \quad \frac{\partial^2 F(A_{k-1})}{\partial a_i \partial a_i} = p(p-1) \int_0^1 w_k(x)\varphi_i(x)\varphi_i(x) dx,$$

where

$$\begin{aligned} \operatorname{sgn}(x, y) &= x && \text{if } y > 0, \\ &= -x && \text{if } y < 0, \\ &= 0 && \text{if } y = 0, \end{aligned}$$

and

$$w_k(x) = |f(x) - L(A_{k-1}, x)|^{p-2}.$$

Substituting (4) and (5) into (3), we have

$$(p-1)H_k \Delta A_k = D_k - H_k A_{k-1},$$

where H_k is an n -by- n matrix whose i, j th entry is

$$\int_0^1 w_k(x)\varphi_i(x)\varphi_j(x) dx,$$

where $\Delta A_k = (\Delta a_1, \Delta a_2, \dots, \Delta a_n)^T$, and where

$$D_k = \left(\int_0^1 w_k(x)f(x)\varphi_1(x) dx, \int_0^1 w_k(x)f(x)\varphi_2(x) dx, \dots, \int_0^1 w_k(x)f(x)\varphi_n(x) dx \right)^T.$$

By noting that $L(B_k, x)$ is the L_{2, w_k} approximation of $f(x)$, we have $H_k B_k = D_k$. Therefore,

$$(p-1)H_k \Delta A_k = H_k(B_k - A_{k-1})$$

and

$$\Delta A_k = \frac{1}{p-1} (B_k - A_{k-1}),$$

or

$$A_k = A_{k-1} + \Delta A_k = \frac{1}{p-1} \{(q-2)A_{k-1} + B_k\}$$

and the above algorithm is a rearranged form of Newton-Raphson's method.

It is noted that the matrix H_k is nonsingular [5], which implies that the convergence of the iteration is quadratic [4] and that the replacement of the operator "integration from 0 to 1" by the operator "summation over a discrete set" does not affect the validity of the algorithm.

3. Acceleration of the Convergence. Convergence of the new method is slow when the starting point A_0 is not sufficiently close to the best approximation, say A^* . As p is increased, more iterations are needed to achieve results of the same accuracy. However, the following acceleration scheme was found to be effective and, in all test cases, the number of iterations required was reduced to one third.

At the k th iteration, use p_k in place of p in the algorithm where

$$p_1 = 2, \quad p_k = \min(p, 2p_{k-1}).$$

We made an analogous modification to the Rice-Usow extension of Lawson's algorithm (LRU algorithm) by replacing p with p_k at the k th iteration and found it to be equally effective. These modified algorithms converge, since p_k becomes p after several iterations.

The new algorithm and LRU method, along with their acceleration schemes, were applied to an example where summation over the equally spaced points was the operator. The results are shown in Table 1.

TABLE 1. *Number of iterations required to approximate $\exp(10x)$, in $[0, 1]$ at equally spaced points, by a polynomial of degree 5, to 4-figure accuracy.*

p	No. of Points	New Method	LRU Method**	LRU Method With Modification
50	30	8	24	8
50	30	26*		
50	50	14	28	9
100	50	20	31	10

* No acceleration scheme was used.

** Rice-Usow extension of Lawson's method.

4. Conclusion. The new algorithm enjoys several advantages over the LRU algorithm. $\{\varphi_i\}$ does not have to be a Chebyshev set, and the algorithm is applicable to the approximation problem for integrals. The iteration does not have to be re-started, and one will not "accidentally" set the weight function to zero at any point.

Consequently, the new algorithm is simpler to program and has wider applications. With the proposed acceleration scheme the convergence seems to be faster. However, the LRU method, with the acceleration scheme proposed in this paper, showed even faster convergence.

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